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Observable Consequences from Second-Type State Vectors of Quantum Mechanics

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Abstract

Starting from a "singlet" state vector for two correlated systems we find an observable whose expectation is 3 according to quantum mechanics, while it has a maximum value of 1 if only state vectors of the first type are considered. This allows a much easier experimental check of the hitherto unobserved state vectors of the second type than suggested by Bell's inequality.

1. *Introduction*

Given two quantum mechanical systems S and T if the complex $\Sigma = S + T$ has a state vector

$$|\eta\rangle = |\psi\rangle \otimes |\phi\rangle \tag{1.1}$$

where $|\psi\rangle$ describes S and $|\phi\rangle$ describes T, we say that $|\eta\rangle$ is a *vector of the first type.* If the vector describing $\Sigma = S + T$ cannot be written under the form (1.1), we say that it is a *vector of the second type.* An example of such a state vector is the singlet state of two spin-$\frac{1}{2}$ particles, which is given by

$$|\eta_{\text{singlet}}\rangle = (1/\sqrt{2})\{|u_+\rangle|v_-\rangle - |u_-\rangle|v_+\rangle\}$$

where $|u_\pm\rangle$ are states for the first particle with third component of the spin equal to $\pm\frac{1}{2}\hbar$, respectively, and $|v_\pm\rangle$ similarly describes the spin of the second particle.

It is easy to show that $|\eta_{\text{singlet}}\rangle$ *cannot* be written under the form (1.1).

The distinction between these two types of state vectors has received increasing attention in recent years (D'Espagnat, 1965; Jammer, 1974; Jauch,

1971) when it has been realized, on the one hand, that most of the unresolved fundamental difficulties of quantum mechanics (EPR-type paradoxes, theory of measurement and "reduction of the wave-packet", unreconciliability with local causality) can be traced back to the existence of state vectors of the *second* type, and, on the other hand, that no *direct* experimental verification of the existence in nature of such vectors exists.

A concept useful to the experimental distinction between the two types of vectors is that of *sensitive observables*, which are defined as those observables whose expectation value over a statistical ensemble of N identical pairs of systems $S + T$ is different according to whether the N pairs are described by a mixture of state vectors of the first type, or by a state vector of the second type (Capasso et al., 1973).

J. S. Bell (1963) has obtained an inequality (see also Clauser et al., 1969; Wigner, 1970), arising solely from the hypothesis of local hidden variables, which can be written

$$|P(ab) - P(ab')| + |P(a'b) + P(a'b')| \leqslant 2 \qquad (1.2)$$

where $P(ab)$ is a correlation function measured when the first (second) apparatus parameter is set to the value $a(b)$. For instance a and b could indicate the direction of the axes of two polarizers.

According to quantum mechanics the left-hand side of (1.2) can be as large as $2\sqrt{2}$, thus showing that quantum mechanics cannot be generalized to a local hidden-variable theory. It has been shown in Capasso et al. (1973) that essentially the left-hand side of (1.2) is a *sensitive observable*, since all state vectors of the first type always satisfy the inequality (1.2).

It has recently been found that a general theorem exists (which we will call *Theorem F*) that allows a much easier experimental test of the existence of state vectors of the second type than the one deduced from Bell's theorem.

2. *A Sensitive Observable for Two-Photon Polarization*

In Fortunato and Selleri (1976) the following theorem has been proved :

> *Theorem F.* If $|\eta\rangle$ is a state vector of the *second* type for two correlated systems S and T the projecton operator
>
> $$\Gamma_\eta \equiv |\eta \times \eta| \qquad (2.1)$$
>
> represents a sensitive observable for the system $S + T$.

We will show in the following that a straightforward application of the previous theorem allows one to suggest an experiment that should provide a new and meaningful test of the quantum mechanical description of two correlated photons.

Two photons with total angular momentum equal to zero are described by the state vector (of the second type)

$$|\eta_0\rangle = \frac{1}{\sqrt{2}}\left\{|x\rangle|y'\rangle - |y\rangle|x'\rangle\right\} \tag{2.2}$$

where $|x\rangle$ and $|y\rangle$ are state vectors for the first photon with linear polarization along the x and y axes, respectively, and $|x'\rangle$, $|y'\rangle$ similarly describe the second photon. From (2.1) one deduces that

$$\Gamma_0 = \tfrac{1}{2}\{|x\rangle\langle x| \otimes |y'\rangle\langle y'| + |y\rangle\langle y| \otimes |x'\rangle\langle x'| - |y\rangle\langle x| \otimes |x'\rangle\langle y'| - |x\rangle\langle y| \otimes |y'\rangle\langle x'|\} \tag{2.3}$$

If one puts

$$|x\rangle\langle x| = \frac{1+R}{2}, \qquad |x\rangle\langle y| = \frac{P+iQ}{2}$$
$$|y\rangle\langle y| = \frac{1-R}{2}, \qquad |y\rangle\langle x| = \frac{P-iQ}{2} \tag{2.4}$$

with P, Q, R Hermitian operators, and if one defines in a similar way P', Q', R' in terms of $|x'\rangle\langle y'|$, and so on, one obtains

$$\Gamma_0 = \tfrac{1}{4}\{1 - P\otimes P' - Q\otimes Q' - R\otimes R'\} \tag{2.5}$$

The physical meaning of the operators P, Q, R and P', Q', R' is easily obtained by noticing that their eigenstates and eigenvalues are given by

$$R|x\rangle = |x\rangle$$
$$R|y\rangle = -|y\rangle \tag{2.6}$$

$$P\left(\frac{|x\rangle + |y\rangle}{\sqrt{2}}\right) = \frac{|x\rangle + |y\rangle}{\sqrt{2}}$$
$$P\left(\frac{|x\rangle - |y\rangle}{\sqrt{2}}\right) = -\frac{|x\rangle - |y\rangle}{\sqrt{2}} \tag{2.7}$$

$$Q\left(\frac{|x\rangle + i|y\rangle}{\sqrt{2}}\right) = \frac{|x\rangle + i|y\rangle}{\sqrt{2}}$$
$$Q\left(\frac{|x\rangle - i|y\rangle}{\sqrt{2}}\right) = -\frac{|x\rangle - i|y\rangle}{\sqrt{2}} \tag{2.8}$$

Therefore the only eigenvalues of P, Q, R are ± 1 and their eigenstates are linear polarization states along x and y (for R), linear polarization states along 45° and −45° (for P), circular polarization states, clockwise and anticlockwise (for Q). Strictly similar interpretations hold for P', Q', R'.

In terms of expectation values one gets from (2.5)

$$\langle\Gamma_0\rangle = \tfrac{1}{4}[1 - \langle\mathscr{P}\mathscr{P}'\rangle - \langle\mathscr{Q}\mathscr{Q}'\rangle - \langle\mathscr{R}\mathscr{R}'\rangle] \tag{2.9}$$

where $\mathscr{P}$ is the polarization observable corresponding to P and so on.

Now the essential point is that the observable Γ_0 is sensitive (Theorem F), which imples that $\langle \Gamma_0 \rangle$ is different if calculated over the state $|\eta_0\rangle$ or over any mixture of state vectors of the first type. In fact

$$\langle \eta_0 | \Gamma_0 | \eta_0 \rangle = 1 \tag{2.10}$$

from the very definition of Γ_0, while if one considers the most general state vector of the first type for two photons :

$$\begin{aligned} |\xi\rangle &= |\gamma\rangle |\gamma'\rangle \\ |\gamma\rangle &= a|x\rangle + b|y\rangle \\ |\gamma'\rangle &= c|x'\rangle + d|y'\rangle \end{aligned} \tag{2.11}$$

with

$$|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1 \tag{2.12}$$

a simple calculation shows that

$$\langle \xi | \Gamma_0 | \xi \rangle = \tfrac{1}{4}[1 - \Delta\Delta' - \sqrt{1-\Delta^2}\sqrt{1-\Delta'^2} \cos(\Phi - \Phi')] \tag{2.13}$$

where

$$\Delta = |a|^2 - |b|^2, \qquad \Delta' = |c|^2 - |d|^2 \tag{2.14}$$

$$\Phi = \frac{1}{i}\ln \frac{a^* b}{|a|\cdot|b|}, \qquad \Phi' = \frac{1}{i}\ln \frac{c^* d}{|c|\cdot|d|} \tag{2.15}$$

The maximum value of $\langle \xi | \Gamma_0 | \xi \rangle$ is easily shown to be obtained when

$$\begin{aligned} &\Phi - \Phi' = n\pi \qquad (n \text{ integer odd}) \\ &\Delta' = -\Delta \end{aligned} \tag{2.16}$$

and when

$$\begin{aligned} &\Phi - \Phi' = n\pi \qquad (n \text{ integer even}) \\ &\Delta' = -\Delta = \pm 1 \end{aligned} \tag{2.16$'$}$$

In both cases one has

$$\langle \xi | \Gamma_0 | \xi \rangle_{max} = \tfrac{1}{2} \tag{2.17}$$

The large difference between (2.10) and (2.17) should allow a rather easy experimental discrimination between state vectors of the first and of the second type.

Incidentally this result makes quantitative a difference that Theorem F states only in a qualitative way.

In the case of a mixture of N identical pairs of photons n_1 of which with state vector (of the first type) $|\xi_1\rangle$, n_2 of which with $|\xi_2\rangle$, and so on ($\Sigma_i n_i = N$)

one obviously has

$$\langle\Gamma_0\rangle = \sum_i \frac{n_i}{N}\langle\xi_i|\Gamma_0|\xi_i\rangle \leqslant \sum_i \frac{n_i}{N}\langle\xi_i|\Gamma_0|\xi_i\rangle_{\max} = \sum_i \frac{n_i}{N}\cdot\frac{1}{2} = \frac{1}{2} \qquad (2.18)$$

Therefore the result (2.17) holds for an arbitrary mixture of vectors of the first type.

3. *Empirical Consequences*

From (2.9), (2.10), and (2.17) it follows

$$-\langle \mathscr{P}\mathscr{P}'\rangle - \langle \mathscr{Q}\mathscr{Q}'\rangle - \langle \mathscr{Q}\mathscr{Q}'\rangle = 3 \qquad (3.1)$$

if the two correlated photons are described by the state vector of the second type $|\eta_0\rangle$, while

$$-\langle \mathscr{P}\mathscr{P}'\rangle - \langle \mathscr{Q}\mathscr{Q}'\rangle - \langle \mathscr{R}\mathscr{R}'\rangle \leqslant 1 \qquad (3.2)$$

if they are described as an arbitrary mixture of state vectors of the first type.

The difference between the two previous relations is similar to the one obtained from Bell's inequality, which is always satisfied by state vectors of the first type but is violated by those of the second type. It is therefore interesting to compare (3.2) with Bell's inequality.

1. The first idea would be to try to deduce (3.2) from local causality. We did not succeed in so doing, but the problem deserves a more careful analysis.

2. The expectation values of products of dichotomic observables $\langle\mathscr{P}\mathscr{P}'\rangle$, $\langle\mathscr{Q}\mathscr{Q}'\rangle$, and $\langle\mathscr{R}\mathscr{R}'\rangle$, entering into (3.2), are strictly analogous to the correlation functions [$P(ab)$, . .] entering into Bell's inequality [see equation (1.2)]. The only difference is that they are calculated for equal values of the parameters a and b entering $P(ab)$. A change of notation perhaps makes clearer these points :One could write

$$\begin{aligned}\langle\mathscr{P}\mathscr{P}'\rangle &= P(45^\circ, 45^\circ)\\ \langle\mathscr{Q}\mathscr{Q}'\rangle &= P(\text{RHC}, \text{RHC})\\ \langle\mathscr{R}\mathscr{R}'\rangle &= P(0^\circ, 0^\circ)\end{aligned} \qquad (3.3)$$

where 0° indicates a transmission measurement of a photon through a polarizer with polarization axis along x; 45° indicates the same with polarization axis at 45°; RHC indicates a transmission measurement of a photon through a right-handed circular polarizer.

3. The difference between the upper limit of (3.2) and the quantum mechanical prediction (3.1) is considerably larger than the similar difference for Bell's inequality. In fact one is comparing a 200% difference with a ~42% difference. Careful consideration of the efficiencies of the polarizers (Horne, 1969), which reduces by about a factor of two the observable gap in Bell's case, should not be so essential in the present case.

4. The different experimental results obtained by Freedman and Clauser (1972) and by Holt and Pipkin (1975) might be easier to reconcile by measuring the correlation functions entering into (3.2).

The present calculation shows, we think, the power of Theorem F, whose consequences we propose to investigate systematically in a following paper.

Just as a more general example, it is easy to show that the sensitive observable corresponding to the state vector

$$|\eta\rangle = \{\alpha |x\rangle |y'\rangle + \beta |y\rangle |x'\rangle\} \tag{3.4}$$

of two photons is

$$\Gamma = \tfrac{1}{4}\{(1 - RR') + \delta(R - R') + R_\delta \cos\varphi(PP' + QQ') + R_\delta \sin\varphi(PQ' - QP')\} \tag{3.5}$$

where

$$R_\delta = \sqrt{1-\delta^2}, \qquad \delta = |\alpha|^2 - |\beta^2|, \qquad \varphi = \frac{1}{i}\ln\frac{\alpha^*\beta}{|\alpha|\cdot|\beta|}$$

and we can also show that its expectation value over the state $|\eta\rangle$ is of course again 1, while its expectation value over a mixture of state vectors of the first type follows from considerations strictly similar to those of the previous section to be no more than $\frac{1}{2}(1 + |\delta|)$.

We consider the present paper a modest contribution along the line of research going from the EPR paradox to Bell's inequality. The time seems to approach when choices about quantum mechanical paradoxes will be made by experiments.

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